Math 5C Extra Credit Problems

A solution to any of these problems is worth 3 perfect quiz scores. Present a complete solution to me in office hours to receive credit. Each of these problems is very difficult and will require perseverance and cleverness to solve. On that note, use absolutely *any* resource you want in figuring out a problem; credit comes from demonstrating understanding to me. Each problem has a solution using ideas from this course, although other methods might exist.

Oh, and these problems are pretty fun.

1. For positive numbers t, let $\{t\}$ denote the fractional (or decimal) part of t. For example, $\{3\} = 0$ and $\{7.5\} = 0.5$. Evaluate
$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx.$$

2. Define the Riemann zeta function as $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$. Evaluate $\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k}$.

3. From one version of the second midterm, we know that
$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1}$$
converges. What is the exact value of the sum?

4. Let R be a bounded region in $\mathbb{R}^2$ with smooth boundary and assume the origin is inside R. Let f be a scalar function defined so that $f = 0$ on ∂R and for every smooth function g,
$$\iint_R g\Delta f \, dA = g(0).$$
In other words, f is the Green function for R with singularity at the origin. Now suppose that the *region* moves in time. At a point on the boundary, the normal velocity V of ∂R is given by
$$V = \frac{\partial f}{\partial n}.$$
As R changes in time, f also changes to preserve the properties in its definition. Given a harmonic function g, prove that
$$\frac{d}{dt} \iint_R g \, dA = g(0).$$
This fact has big implications in mathematical physics.

5. Let R be a nice region in $\mathbb{C}$. Learn about Stokes' theorem in this case:
$$\iint_R \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz = \oint_{\partial R} f \, dz.$$
Use this formula to prove the Cauchy integral theorem: if $f : \mathbb{C} \to \mathbb{C}$ is analytic, $\displaystyle\int_{\partial R} f \, dz = 0$.